Homework 10 Solutions

1. Let $A = U\Sigma V^T$ be an SVD of a matrix A. Let $\mathbf{u}_i$ denote the ith column of U and let $\mathbf{v}_i$ denote the ith column of V. In class, we showed that $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$ for all $1 \leq i \leq r$ and $A\mathbf{v}_i = 0$ for all $r+1 \leq i \leq n$.

   (a) Show that $A^T\mathbf{u}_i = \sigma_i \mathbf{v}_i$ for all $1 \leq i \leq r$.

   Taking the transpose of $A = U\Sigma V^T$ gives $A^T = (U\Sigma V^T)^T = V\Sigma^T U^T$. Since U is orthogonal, $U^T U = I$; hence if we multiply $A^T = V\Sigma^T U^T$ by U on the right, we obtain $A^T U = V\Sigma^T$. Notice that if $1 \leq i \leq r$, then the ith column of $A^T U$ is $A\mathbf{u}_i$ and the ith column of $V\Sigma^T$ is $\sigma_i \mathbf{v}_i$. Since the matrices are the same, we have $A^T\mathbf{u}_i = \sigma_i \mathbf{v}_i$.

   (b) Show that $A^T\mathbf{u}_i = 0$ for $r+1 \leq i \leq m$.

   As above, we have $A^T U = V\Sigma^T$. If $r+1 \leq i \leq m$, then the ith column of $V\Sigma^T$ is $\mathbf{0}$ (since the only nonzero entries of Σ^T are the first r diagonal entries); hence $A^T\mathbf{u}_i = 0$. Alternatively, since $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$ for all $1 \leq j \leq r$, we have that $\mathbf{u}_j = A(\mathbf{v}_j/\sigma_j) \in C(A)$. For any $i \geq r+1$, since $\mathbf{u}_i$ is orthogonal to $\mathbf{u}_j$ for all $j \neq i$, it follows that $\mathbf{u}_i \in C(A)^\perp = N(A^T)$. Hence $A^T\mathbf{u}_i = 0$ for all $r+1 \leq i \leq m$.

2. Consider the following SVD of a square matrix A:

$$U\Sigma V^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \end{bmatrix} \begin{bmatrix} -2/3 & \sqrt{3}/3 & -\sqrt{2}/6 & -\sqrt{6}/6 \\ -1/3 & 0 & 2\sqrt{2}/3 & 0 \\ 2/3 & \sqrt{3}/3 & \sqrt{2}/6 & -\sqrt{6}/6 \\ 0 & \sqrt{3}/3 & 0 & \sqrt{6}/3 \end{bmatrix}^T$$

   (a) Determine the eigenvalues of $A^T A$? Explain your reasoning.

   The nonzero eigenvalues of $A^T A$ are the squares of the singular values of A; hence they are $3^2 = 9$ and $\sqrt{3}^2 = 3$. It remains to determine whether $A^T A$ has a zero eigenvalue. Since Σ is 2×4, so is A; hence $A^T A$ is 4×4 and has four real eigenvalues (counted with algebraic multiplicity). Since 3 and $\sqrt{3}$ only appear once in Σ, they each have algebraic multiplicity 1. Thus 0 must also be an eigenvalue.

   So the eigenvalues of $A^T A$ are $0, 3, 9$.

   (b) What is the rank of A? Explain your reasoning.

The rank of A is the number of singular values of A. Since there are two singular values, rank$A = 2$.

(c) Determine orthonormal bases of the four fundamental subspaces of A. Explain your reasoning.

Since A is 2×4 with rank 2, the dimensions of the four fundamental subspaces are: $\dim C(A) = 2, \dim C(A^T) = 2, \dim N(A) = 4 - 2 = 2$, and $\dim N(A^T) = 2 - 2 = 0$. Hence the left nullspace has no basis. The bases of the other three subspaces are as follows: the first two columns of V form an orthonormal basis for $C(A^T)$, the second two columns of V form an orthonormal basis for $N(A^T)$, and the first two columns of U form an orthonormal basis for $C(A)$.

(d) Diagonalize $A^T A$. Explain your reasoning.

Since U is orthogonal, $U^T U = I$. Moreover, since V is orthogonal, $V^T = V^{-1}$. Hence

$$A^T A = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma^T U^T U \Sigma V^T = V(\Sigma^T \Sigma)V^T = V(\Sigma^T \Sigma)V^{-1}.$$

Now since $\Sigma^T \Sigma = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is diagonal, $A^T A = V(\Sigma^T \Sigma)V^{-1}$ is a diagonalization of $A^T A$.

(e) Determine the SVD rank 1 approximation of A. Show your work.

The SVD rank 1 approximation is

$$A_1 = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T = 3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} -2/3 & -1/3 & 2/3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix}.$$

(f) What is the largest possible value of $||A\mathbf{u}||$, where $\mathbf{u}$ is a unit vector? Explain your reasoning.

We know that $||A\mathbf{u}||$ is maximized when $\mathbf{u} = \mathbf{v}_1$, the first singular vector. By problem 1, $A\mathbf{v}_1 = \sigma_1 \mathbf{u}_1$. Since $\mathbf{u}_1$ is a unit vector, we have $||A\mathbf{v}_1|| = 3||\mathbf{u}_1|| = 3$.

3. Consider the following data points.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

(a) Describe how to find the SVD 2-dimensional best fit subspace of the above data points (no computation required).

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and let $A = U\Sigma V^T$ be an SVD. Then the $2-$dimensional best fit subspace is the span of the first two columns of V.

(b) Use Julia to find a basis for the 2-dimensional best fit subspace of the above data points. Include a screen shot of your code and output.

Problem 3b

```
In [2]: using LinearAlgebra

        A = [1 2 3 ; 4 5 6 ; 7 8 9]
        U,s,V = svd(A, full = true)
        B=[V[:,1] V[:,2]] #Matrix whose columns form the basis

Out[2]: 3×2 Matrix{Float64}:
         -0.479671  -0.776691
         -0.572368  -0.0756865
         -0.665064   0.625318
```

4. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 5 \\ 3 & 5 & 9 \end{bmatrix}$. Use Julia for the following calculations. Find an SVD for A and then compute the rank 1, 2, and 3 approximations for A and determine their Frobenius distances from A. Include a screen shot of your code and output.

[the command "norm(M)" provides the Frobenius norm of a matrix M]

Problem 4

In [3]:
```
A = [1 1 1; 1 2 3; 1 2 5; 3 5 9]
U,s,V = svd(A, full = true)
```

Out[3]:
```
SVD{Float64, Float64, Matrix{Float64}, Vector{Float64}}
U factor:
4×4 Matrix{Float64}:
 -0.124027   0.65058   -0.557999   -0.5
 -0.293926   0.22997    0.781486   -0.5
 -0.427766  -0.701211  -0.274444   -0.5
 -0.845719   0.179339  -0.0509567   0.5
singular values:
3-element Vector{Float64}:
 12.67798964012828
  1.088366683172887
  0.28989075142074394
Vt factor:
3×3 Matrix{Float64}:
 -0.266831  -0.457171  -0.848408
  0.659113   0.55569   -0.506734
 -0.703116   0.694409  -0.153052
```

In [10]:
```
A1 = s[1]*U[:,1]*V[:,1]' #rank 1 approximation
```

Out[10]:
```
4×3 Matrix{Float64}:
 0.419567  0.71886  1.33404
 0.994317  1.7036   3.1615
 1.44708   2.47933  4.6011
 2.86096   4.90179  9.09665
```

In [11]:
```
norm(A1-A) #Frobenius distance
```

Out[11]:
```
1.1263119837771585
```

In [12]:
```
A2 = s[1]*U[:,1]*V[:,1]'+ s[2]*U[:,2]*V[:,2]' #rank 2 approximation
```

Out[12]:
```
4×3 Matrix{Float64}:
 0.886265  1.11233  0.975242
 1.15929   1.84268  3.03467
 0.944061  2.05525  4.98782
 2.98961   5.01026  8.99774
```

In [13]:
```
norm(A2-A) #Frobenius distance
```

Out[13]:
```
0.2898907514207438
```

In [14]:
```
A3 = s[1]*U[:,1]*V[:,1]'+ s[2]*U[:,2]*V[:,2]'+s[3]*U[:,3]*V[:,3]' #rank 3 approximation
```

Out[14]:
```
4×3 Matrix{Float64}:
 1.0  1.0  1.0
 1.0  2.0  3.0
 1.0  2.0  5.0
 3.0  5.0  9.0
```

In [15]:
```
norm(A3-A) #Frobenius distance
```

Out[15]:
```
5.545561229666063e-15
```

In [ ]:

5. Suppose A is a 2×2 symmetric matrix with orthogonal diagonalization

$$A = Q \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} Q^T, \quad \text{where } Q = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}.$$

Find an SVD of A.

Since A is symmetric, $A^T = A$ and so $A^T A = A^2$. Thus the eigenvalues of $A^T A$ are $\lambda_1^2 = 3^2 = 9$ and $\lambda_2^2 = (-2)^2 = 4$. It follows that the singular values of A are $\sigma_1 = 3$ and $\sigma_2 = 2$. Thus we have $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Recall that the eigenvectors of A are also eigenvectors of A^2 since if $A\mathbf{v} = \lambda\mathbf{v}$, then $A^2\mathbf{v} = \lambda^2\mathbf{v}$. Hence the columns of Q, which are orthonormal eigenvectors of A are also orthonormal eigenvectors of $A^2 = A^T A$. Hence we can set $V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$. Now the columns of U are $\dfrac{A\mathbf{v}_1}{\sigma_1} = \dfrac{3\mathbf{v}_1}{3} = \mathbf{v}_1$ and $\dfrac{A\mathbf{v}_2}{\sigma_2} = \dfrac{-2\mathbf{v}_2}{2} = -\mathbf{v}_2$. Hence $U = \begin{bmatrix} \mathbf{v}_1 & -\mathbf{v}_2 \end{bmatrix}$. So we have the SVD

$$A = \begin{bmatrix} \mathbf{v}_1 & -\mathbf{v}_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^T$$

6. Show that if the SVD of a square matrix A is also a diagonalization of A, then A is symmetric and positive semi-definite.

[Note: the converse is also true!]

Let $A = U\Sigma V^T$ be an SVD of an $n \times n$ matrix A of rank r that is also a diagonalization. Then Σ is a diagonal matrix containing the eigenvalues of A and $U = V$ contains orthonormal eigenvectors of A corresponding to the eigenvalues. Since U is orthogonal, this provides an orthogonal diagonalization, which implies that A is symmetric. Moreover, since the diagonal entries of Σ are nonnegative, it follows that the eigenvalues of A are nonnegative and so A is positive semi-definite.